EE 508
Lecture 7

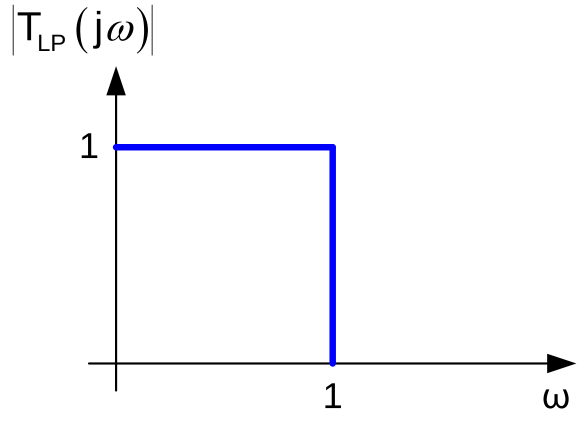
The Approximation Problem

Review from Last Time

The Approximation Problem

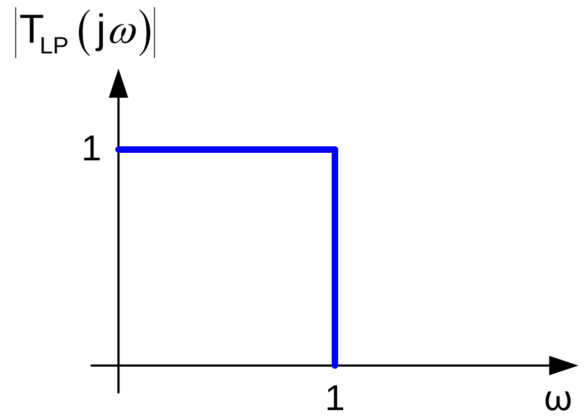
The goal in the approximation problem is simple, just want a function $T_A(s)$ or $H_A(z)$ that meets the filter requirements.

Will focus primarily on approximations of the standard normalized lowpass function



- Frequency scaling will be used to obtain other LP band edges
- Frequency transformations will be used to obtain HP, BP, and BR responses

The Approximation Problem



$$T_A(s) = ?$$

$T_A(s)$ is a rational fraction in s

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

Rational fractions in s have no discontinuities in either magnitude or phase response

No natural metrics for $T_A(s)$ that relate to magnitude and phase characteristics (difficult to meaningfully compare $T_{A1}(s)$ and $T_{A2}(s)$)

Review from Last Time

Magnitude Squared Approximating Functions

$$T(s)=\frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

$$T(j\omega)=\frac{\left[F_1\left(\omega^2\right)\right]+j\left[\omega F_2\left(\omega^2\right)\right]}{\left[F_3\left(\omega^2\right)\right]+j\left[\omega F_4\left(\omega^2\right)\right]}$$

$$\left|T(j\omega)\right|=\sqrt{\frac{\left[F_1\left(\omega^2\right)\right]^2+\omega^2\left[F_2\left(\omega^2\right)\right]^2}{\left[F_3\left(\omega^2\right)\right]^2+\omega^2\left[F_4\left(\omega^2\right)\right]^2}}$$

Thus $\left|T(j\omega)\right|$ is an even function of ω

It follows that $\left|T(j\omega)\right|^2$ is a rational fraction in ω^2 with real coefficients

Since $\left|T(j\omega)\right|^2$ is a real variable, natural metrics exist for comparing approximating functions to $\left|T(j\omega)\right|^2$

Review from Last Time

Magnitude Squared Approximating Functions

$$T(s)=\frac{\sum_{i=0}^{m}a_is^i}{\sum_{i=0}^{n}b_is^i}$$

If a desired magnitude response is given, it is common to find a rational fraction in ω^2 with real coefficients, denoted as $H_A(\omega^2)$, that approximates the desired magnitude squared response and then obtain a function $T_A(s)$ that satisfies the relationship $\left|T_A(j\omega)\right|^2=H_A(\omega^2)$

$H_A(\omega^2)$ is real so natural metrics exist for obtaining $H_A(\omega^2)$

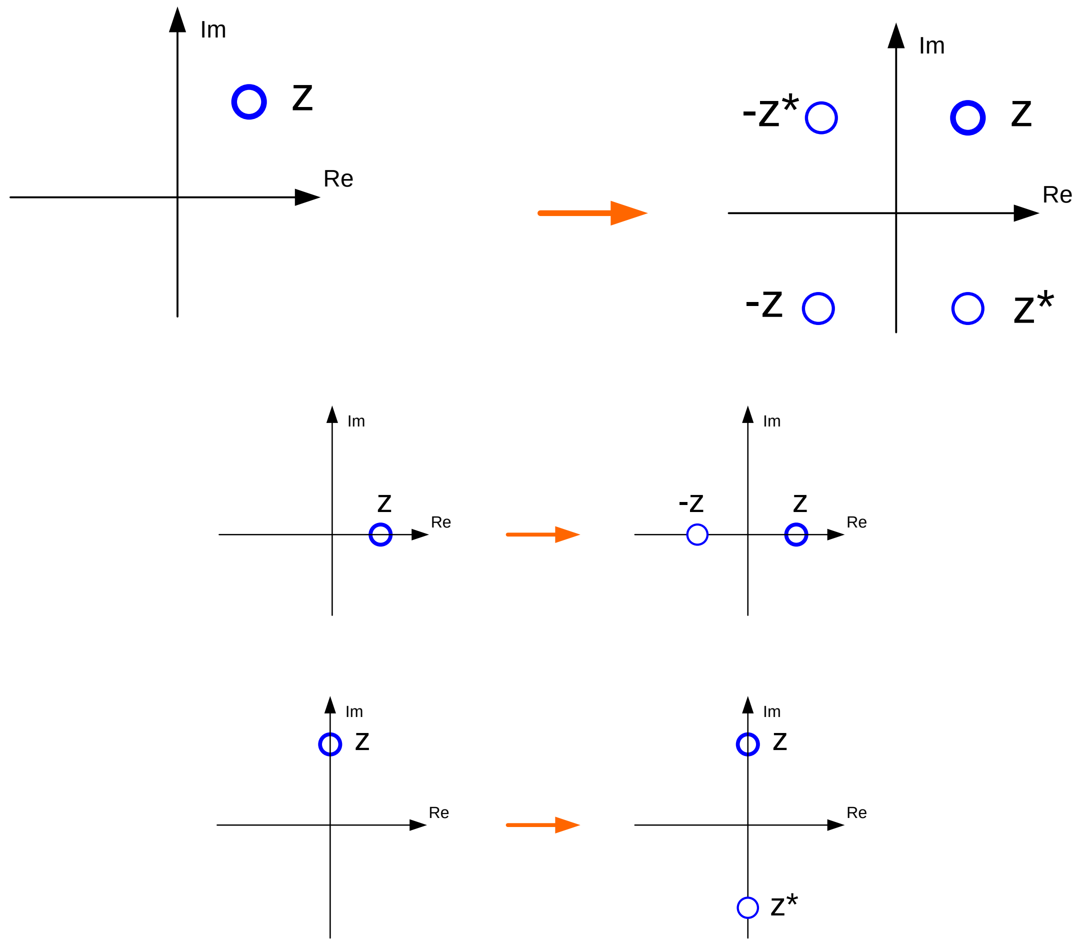
$$H_A(\omega^2)=\frac{\sum_{i=0}^{2l}c_i\omega^{2i}}{\sum_{i=0}^{2k}d_i\omega^{2i}}$$

Obtaining $T_A(s)$ from $H_A(\omega^2)$ is termed the inverse mapping problem

But how is $T_A(s)$ obtained from $H_A(\omega^2)$?

Review from Last Time

Observation: If z is a zero (pole) of $H_A(\omega^2)$, then $-z$, z^*, and $-z^*$ are also zeros (poles) of $H_A(\omega^2)$



Thus, roots come as quadruples if off of the axis and as pairs if they lay on the axis

Magnitude Squared Approximating Functions

If a desired magnitude response is given, it is common to find a rational fraction in ω^2 with real coefficients, denoted as $H_A(\omega^2)$, that approximates the desired magnitude squared response and then obtain a function $T_A(s)$ that satisfies the relationship $\left|T_A(j\omega)\right|^2 = H_A\left(\omega^2\right)$

Inverse mapping may not exist !

To make this approach practical it is essential that a method be developed for determining if an inverse mapping exists and, if it exists, to determine **an** inverse mapping!

Inverse MappingTheorem: If $H_A(\omega^2)$ is a rational fraction with real coefficients with no poles or zeros of odd multiplicity on the real axis, then there exists a real number H_0 such that the function

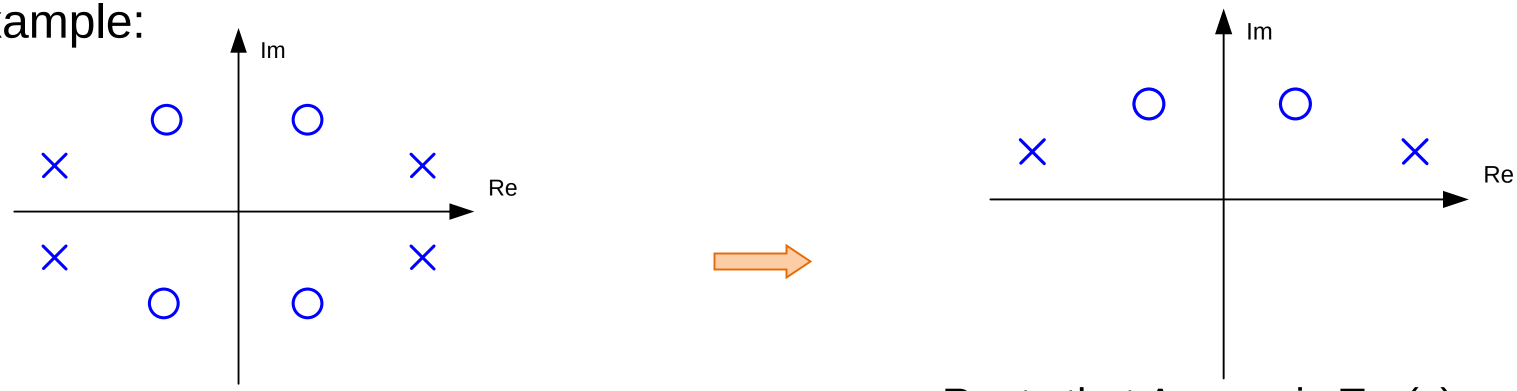
$$T_{AM}(s)=\frac{H_0(s\text{-}jz_1)(s\text{-}jz_2)\bullet\ldots\bullet(s\text{-}jz_m)}{(s\text{-}jp_1)(s\text{-}jp_2)\bullet\ldots\bullet(s\text{-}jp_n)}$$

is a minimum phase rational fraction with real coefficents that satisfies the relationship

$$\left|T_{AM}(j\omega)\right|=\sqrt{H_A(\omega^2)}$$

where $\{z_1, z_2, \ldots z_m\}$ are the upper half-plane zeros of $H_A(\omega^2)$ and exactly half of the real axis zeros,
and where where $\{p_1, p_2, \ldots p_n\}$ are the upper half-plane poles of $H_A(\omega^2)$ and exactly half of the real axis poles.

Example:



Roots of $H_A(\omega^2)$

Roots that Appear in $T_{AM}(s)$

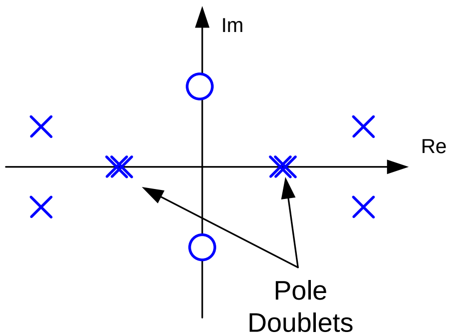
(but multiplied by j)

$$H_A\left(\omega^2\right)=\frac{H_0^2\left[(\omega-z_1)(\omega-z_2)\bullet\ldots\bullet(\omega-z_m)\right]\bullet\left[(\omega+z_1)(\omega+z_2)\bullet\ldots\bullet(\omega+z_m)\right]}{\left[(\omega-p_1)(\omega-p_2)\bullet\ldots\bullet(\omega-p_n)\right]\bullet\left[(\omega+p_1)(\omega+p_2)\bullet\ldots\bullet(\omega+p_n)\right]}$$

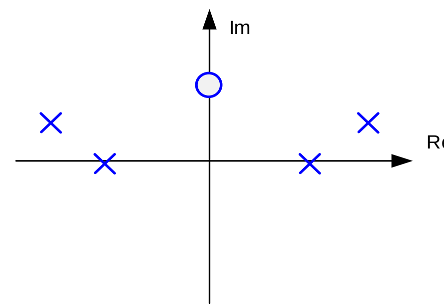
If inverse exists

$$T_{AM}(s)=\frac{H_0(s-jz_1)(s-jz_2)\bullet\ldots\bullet(s-jz_m)}{(s-jp_1)(s-jp_2)\bullet\ldots\bullet(s-jp_n)}$$

Example:

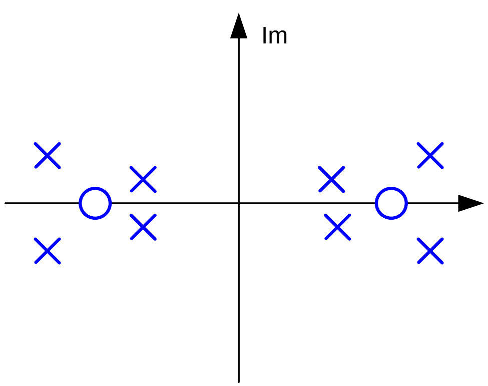


Roots of $H_A(\omega^2)$



Roots that appear in $T_{AM}(s)$

Example:

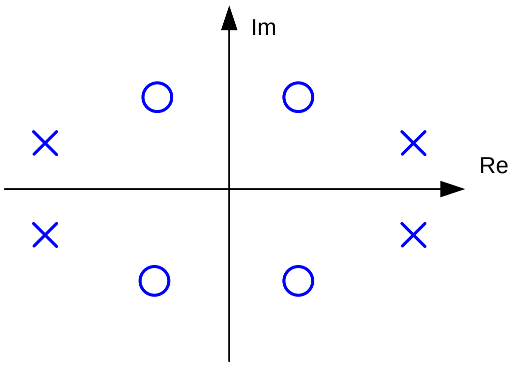


Inverse does not exist because zeros are of odd multiplicity on the real axis

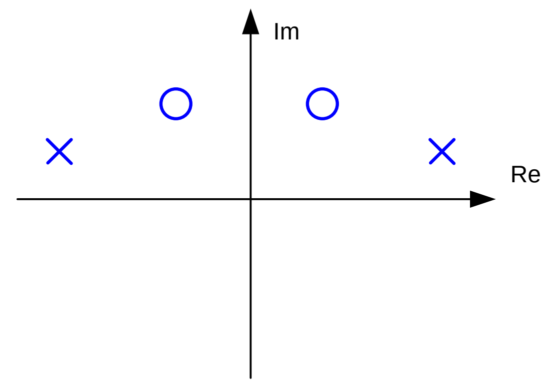
$$H_A\left(\omega^2\right)=\frac{H_0^2\left[\left(\omega\text{-}z_1\right)\left(\omega\text{-}z_2\right)\bullet\ldots\bullet\left(\omega\text{-}z_m\right)\right]\bullet\left[\left(\omega+z_1\right)\left(\omega+z_2\right)\bullet\ldots\bullet\left(\omega+z_m\right)\right]}{\left[\left(\omega\text{-}p_1\right)\left(\omega\text{-}p_2\right)\bullet\ldots\bullet\left(\omega\text{-}p_n\right)\right]\bullet\left[\left(\omega+p_1\right)\left(\omega+p_2\right)\bullet\ldots\bullet\left(\omega+p_n\right)\right]}$$

If inverse exists

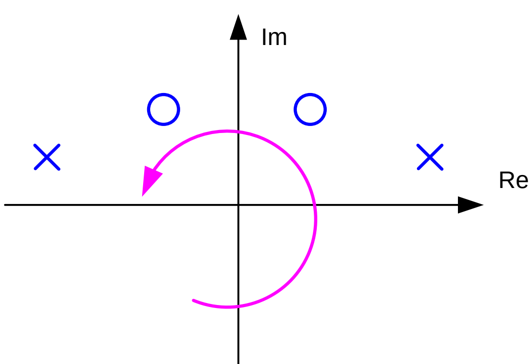
$$T_{AM}\left(s\right)=\frac{H_0\left(s\text{-}jz_1\right)\left(s\text{-}jz_2\right)\bullet\ldots\bullet\left(s\text{-}jz_m\right)}{\left(s\text{-}jp_1\right)\left(s\text{-}jp_2\right)\bullet\ldots\bullet\left(s\text{-}jp_n\right)}$$



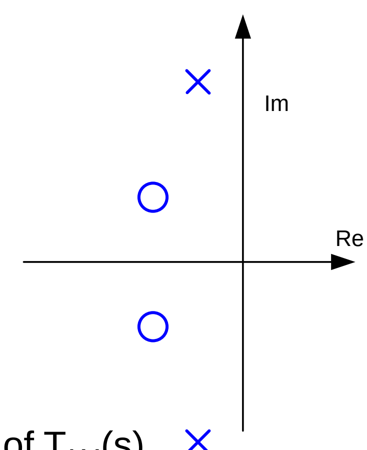
Roots of $H_A(\omega^2)$



Roots that appear in $T_{AM}(s)$



Rotate roots by 90^o



Roots of $T_{AM}(s)$

$$H_A\left(\omega^2\right)=\frac{H_0^2\left[(\omega\text{-}z_1)(\omega\text{-}z_2)\bullet\ldots\bullet(\omega\text{-}z_m)\right]\bullet\left[(\omega+z_1)(\omega+z_2)\bullet\ldots\bullet(\omega+z_m)\right]}{\left[(\omega\text{-}p_1)(\omega\text{-}p_2)\bullet\ldots\bullet(\omega\text{-}p_n)\right]\bullet\left[(\omega+p_1)(\omega+p_2)\bullet\ldots\bullet(\omega+p_n)\right]}$$

$$\Big\Downarrow \text{ If inverse exists}$$

$$T_{AM}(s)=\frac{H_0(s\text{-}jz_1)(s\text{-}jz_2)\bullet\ldots\bullet(s\text{-}jz_m)}{(s\text{-}jp_1)(s\text{-}jp_2)\bullet\ldots\bullet(s\text{-}jp_n)}$$

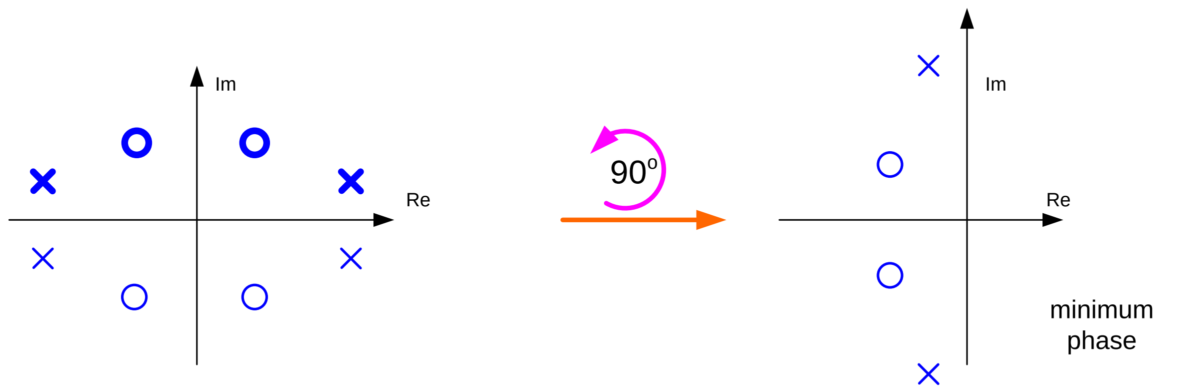
Observations:

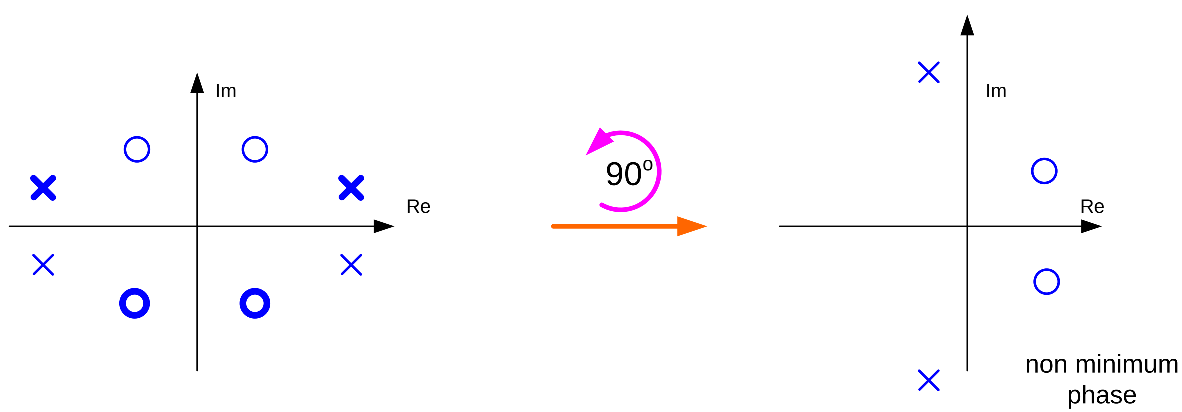
- Coefficients of $T_{AM}(s)$ are real
- If x is a root of $H_A(\omega^2)$, then jx is a root of $T_{AM}(s)$
- Multiplying a root by j is equivalent to rotating it by 90° cc in the complex plane
- Roots of $T_{AM}(s)$ are obtained from roots of $H_A(\omega^2)$ by multiplying by j
- Roots of $T_{AM}(s)$ are upper half-plane roots and exactly half of real axis roots all rotated cc by 90°
- If a root of $H_A(\omega^2)$ has odd multiplicity on the real axis, the inverse mapping does not exist
- Other (often many) inverse mappings exist but are not minimum phase
  (These can be obtained by reflecting any subset of the zeros or poles around the imaginary axis into the RHP)

$$H_A\left(\omega^2\right)=\frac{H_0^2\left[\left(\omega-z_1\right)\left(\omega-z_2\right)\bullet\ldots\bullet\left(\omega-z_m\right)\right]\bullet\left[\left(\omega+z_1\right)\left(\omega+z_2\right)\bullet\ldots\bullet\left(\omega+z_m\right)\right]}{\left[\left(\omega-p_1\right)\left(\omega-p_2\right)\bullet\ldots\bullet\left(\omega-p_n\right)\right]\bullet\left[\left(\omega+p_1\right)\left(\omega+p_2\right)\bullet\ldots\bullet\left(\omega+p_n\right)\right]}$$

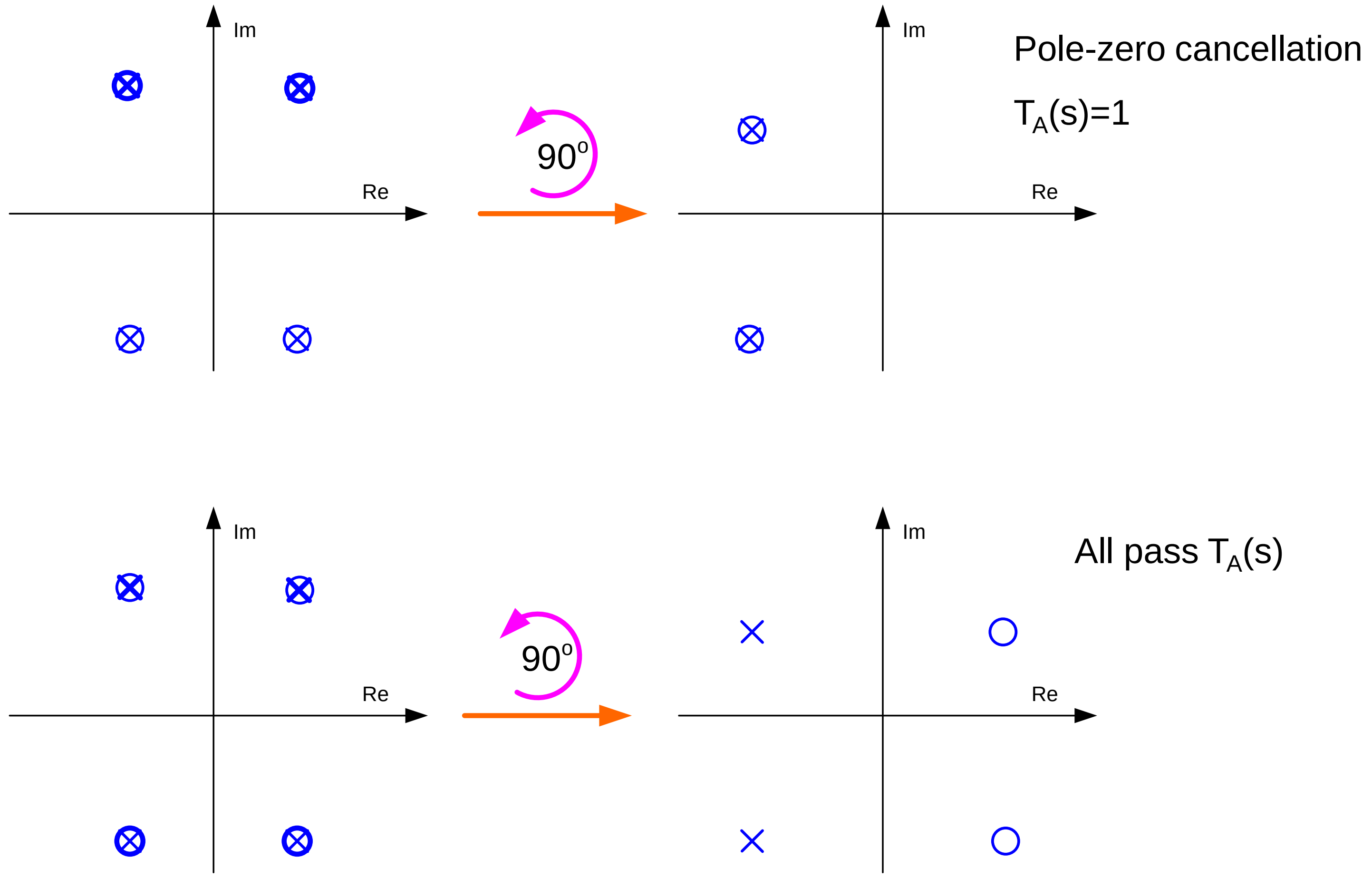
If inverse exists

$$T_{AM}\left(s\right)=\frac{H_0\left(s-jz_1\right)\left(s-jz_2\right)\bullet\ldots\bullet\left(s-jz_m\right)}{\left(s-jp_1\right)\left(s-jp_2\right)\bullet\ldots\bullet\left(s-jp_n\right)}$$



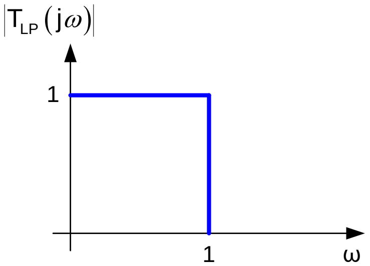


All pass functions (and factors)



- Must not allow cancellations to take place in $H_A(\omega^2)$ to obtain all-pass $T_A(s)$
- Must keep upper HP poles and lower HP zeros in $H_A(\omega^2)$ to obtain all-pass $T_A(s)$
- All-pass $T_A(s)$ is not minimum phase

The Approximation Problem

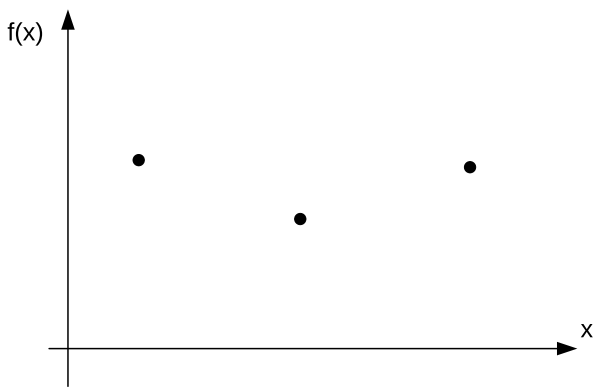


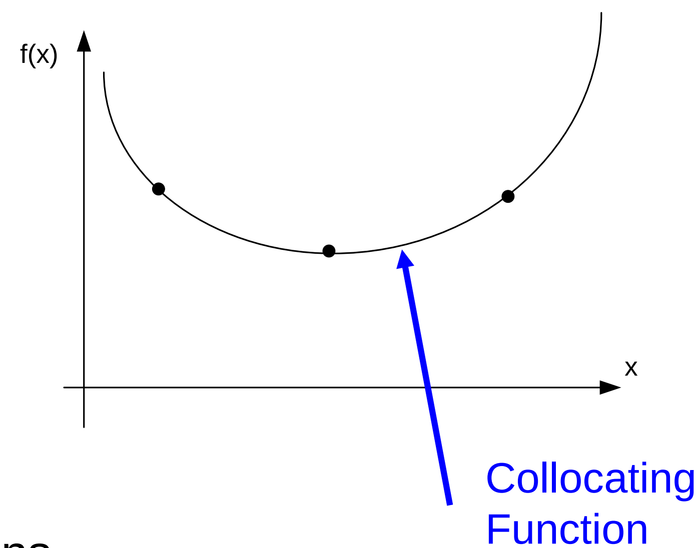
Approach we will follow:

- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares
- Pade Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - $\rightarrow$ Butterworth (BW)
  - $\rightarrow$ Chebyschev (CC)
  - $\rightarrow$ Elliptic
  - $\rightarrow$ Thompson

Collocation

Collocation is the fitting of a function to a set of points (or measurements) so that the function agrees with the sample at each point in the set.





Often consider critically constrained functions

The function that is of interest for using collocation when addressing the approximation problem is $H_A(\omega^2)$

Collocation

Example: Collocation points $\{(x_1,y_1), (x_2,y_2),(x_3,y_3)\}$

Polynomial collocating function (critically constrained)

$$f(x) = a_0 + a_1x + a_2x^2$$

Unknowns: $\{a_1,a_2,a_3\}$

Set of equations:

$$y_1 = a_0 + a_1x_1 + a_2x_1^2$$
$$y_2 = a_0 + a_1x_2 + a_2x_2^2$$
$$y_3 = a_0 + a_1x_3 + a_2x_3^2$$

These equations are linear in the unknowns $\{a_1,a_2,a_3\}$

Can be expressed in matrix form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \bullet \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{X} \bullet \mathbf{A}$$

Solution:

$$\mathbf{A} = \mathbf{X}^{-1} \bullet \mathbf{Y}$$

Closed form solution exists when collocating to a polynomial

Collocation

Is it possible to get a closed-form solution when collocating to a rational fraction?

$$\{(x_1,y_1),(x_2,y_2)\ldots(x_k,y_k)\}$$

$$f(x)=\frac{a_0+a_1x+a_2x^2+\ldots+a_mx^m}{1+b_1x+b_2x^2+\ldots+b_nx^n}$$

where k=m+n+1

The rational fraction is nonlinear in x !

$$y_1\left(1+b_1x_1+b_2x_1^2+\ldots+b_nx_1^n\right)=a_0+a_1x_1+a_2x_1^2+\ldots+a_mx_1^n$$

This can be expressed as

$$y_1=a_0+a_1x_1+a_2x_1^2+\ldots+a_mx_1^n-b_1x_1y_1-b_2x_1^2y_1-\ldots-b_nx_1^ny_1$$

Note this equation is linear in the unknowns $\{a_0,a_1,\ldots a_m,b_1,b_2,\ldots b_n\}$

Collocation

Is it possible to get a closed-form solution when collocating to a rational fraction?

$$\{(x_1, y_1), (x_2, y_2) \ldots (x_k, y_k)\}$$

$$f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m}{1 + b_1 x + b_2 x^2 + \ldots + b_n x^n}$$

where k=m+n+1

$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_m x_1^m - b_1 x_1 y_1 - b_2 x_1^2 y_1 - \ldots - b_n x_1^n y_1$$

$$y_2 = a_0 + a_1 x_2 + a_2 x_2^2 + \ldots + a_m x_2^m - b_1 x_2 y_2 - b_2 x_2^2 y_2 - \ldots - b_n x_2^n y_2$$

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$$y_k = a_0 + a_1 x_k + a_2 x_k^2 + \ldots + a_m x_k^m - b_1 x_k y_k - b_2 x_k^2 y_k - \ldots - b_n x_k^n y_k$$

Collocation

Is it possible to get a closed-form solution when collocating to a rational fraction?

$$\{(x_1, y_1), (x_2, y_2) \ldots (x_k, y_k)\}$$

$$f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m}{1 + b_1 x + b_2 x^2 + \ldots + b_n x^n}$$

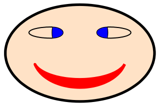
$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_m x_1^m - b_1 x_1 y_1 - b_2 x_1^2 y_1 - \ldots - b_n x_1^n y_1$$
$$y_2 = a_0 + a_1 x_2 + a_2 x_2^2 + \ldots + a_m x_2^m - b_1 x_2 y_2 - b_2 x_2^2 y_2 - \ldots - b_n x_2^n y_2$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$y_k = a_0 + a_1 x_k + a_2 x_k^2 + \ldots + a_m x_k^m - b_1 x_k y_k - b_2 x_k^2 y_k - \ldots - b_n x_k^n y_k$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \bullet \\ \bullet \\ y_k \end{bmatrix} = \begin{bmatrix} 1 \; x_1 \;\; x_1^2 \ldots \; x_1^m - x_1 y_1 - x_1^2 y_1 - \ldots - x_1^n y_1 \\ 1 \; x_2 \;\; x_2^2 \ldots \; x_2^m - x_2 y_2 - x_2^2 y_2 - \ldots - x_2^n y_2 \\ \bullet \\ \bullet \\ 1 \; x_k \;\; x_k^2 \ldots \; x_k^m - x_k y_k - x_k^2 y_k - \ldots - x_k^n y_k \end{bmatrix} \bullet \begin{bmatrix} a_0 \\ a_1 \\ .. \\ a_m \\ b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$$

$$\mathbf{Y = Z \bullet C}$$

$$\mathbf{C = Z^{-1} \bullet Y}$$

Closed form solution when collocating to a rational fraction !



Collocation

Applying to $H_A\left(\omega^2\right)$

$$\left\{(\omega_1, y_1), (\omega_2, y_2) \ldots (\omega_k, y_k)\right\}$$

$$H_A\left(\omega^2\right) = \frac{a_0 + a_1\omega^2 + a_2\omega^4 + \ldots + a_m\omega^{2m}}{1 + b_1\omega^2 + b_2\omega^4 + \ldots + b_n\omega^{2n}}$$

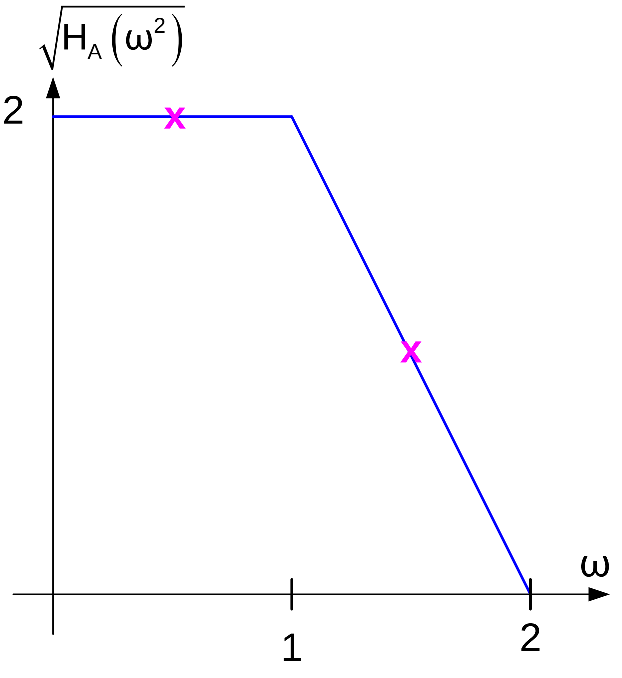
$$\begin{bmatrix} y_1 \\ y_2 \\ \bullet \\ \bullet \\ y_k \end{bmatrix} = \begin{bmatrix} 1 & \omega_1^2 & \omega_1^4 \ldots & \omega_1^{2m} & -\omega_1^2 y_1 & -\omega_1^4 y_1 & -\ldots & -\omega_1^{2n} y_1 \\ 1 & \omega_2^2 & \omega_2^4 \ldots & \omega_2^{2m} & -\omega_2^2 y_1 & -\omega_2^4 y_1 & -\ldots & -\omega_2^{2n} y_1 \\ \bullet \\ \bullet \\ 1 & \omega_k^2 & \omega_k^4 \ldots & \omega_k^{2m} & -\omega_k^2 y_1 & -\omega_k^4 y_1 & -\ldots & -\omega_k^{2n} y_1 \end{bmatrix} \bullet \begin{bmatrix} a_0 \\ a_1 \\ .. \\ a_m \\ b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{Z} \bullet \mathbf{C}$$

$$\mathbf{C} = \mathbf{Z}^{-1} \bullet \mathbf{Y}$$

Collocation

Example:



x denotes collocation points

$$H_A(\omega^2) = \frac{a_0}{1+b_1\omega^2}$$

$$\left.\begin{array}{l} 4 = \dfrac{a_0}{1+b_1\left(\frac{1}{2}\right)^2} \\[2ex] 1 = \dfrac{a_0}{1+b_1\left(\frac{3}{2}\right)^2} \end{array}\right\} \Rightarrow H_A(\omega^2) = \frac{\frac{32}{5}}{1+\left(\frac{12}{5}\right)\omega^2}$$

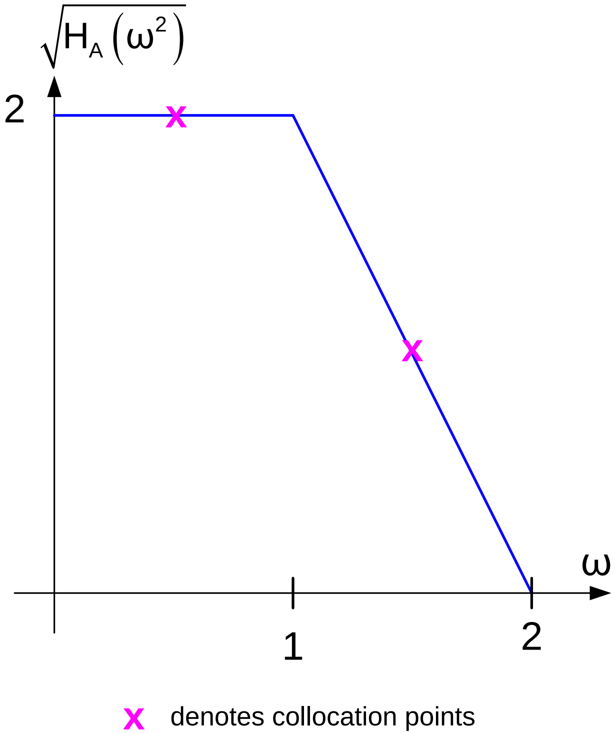
poles at $s = \pm j\sqrt{\frac{5}{12}}$

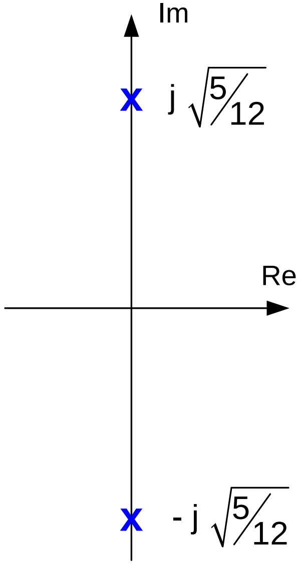
Collocation

Example:

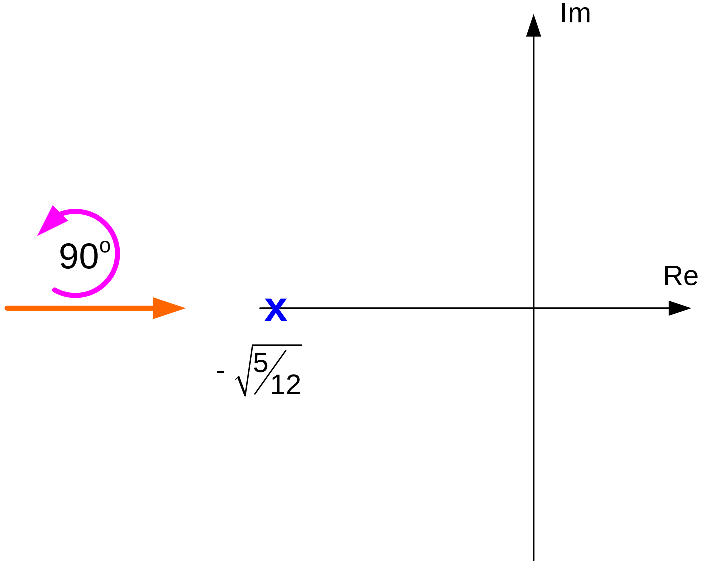
poles at

$$s = \pm j\sqrt{5/12}$$





Roots of $H_A(\omega^2)$



Roots of $T_{AM}(s)$

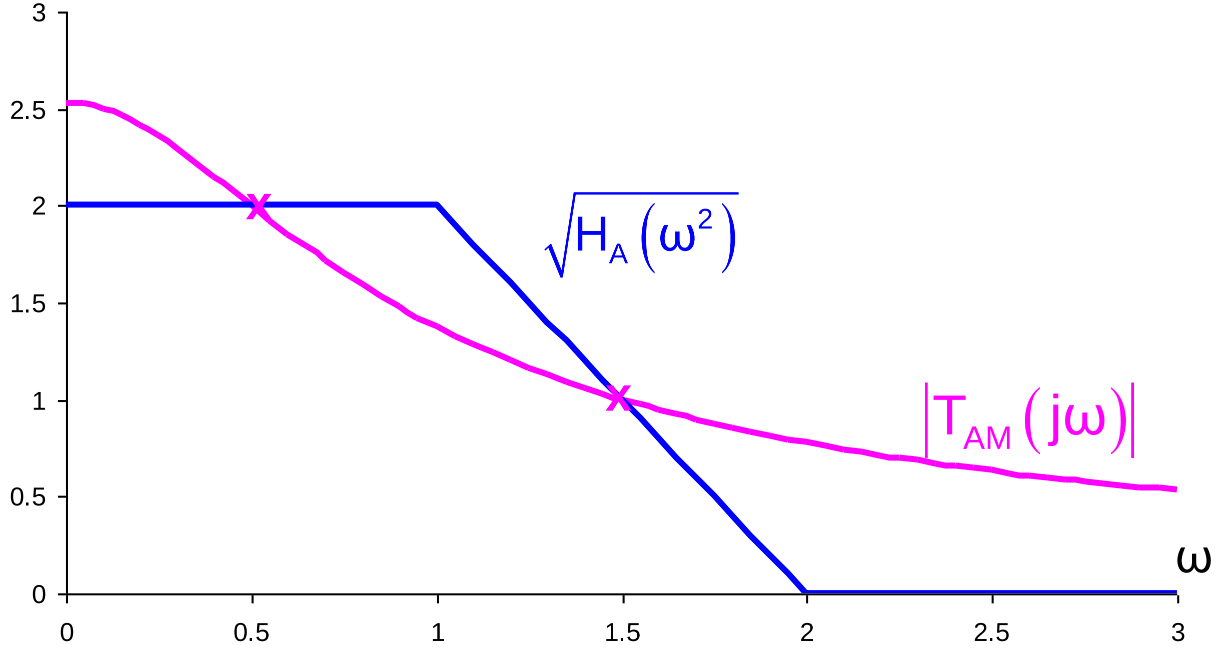
$$T_{AM}(s) = \frac{\sqrt{8/3}}{s + \sqrt{5/12}}$$

Collocation

Example:

$$H_A\left(\omega^2\right) = \frac{32/5}{1 + \left(12/5\right)\omega^2}$$

$$T_{AM}(s) = \frac{\sqrt{8/3}}{s + \sqrt{5/12}}$$



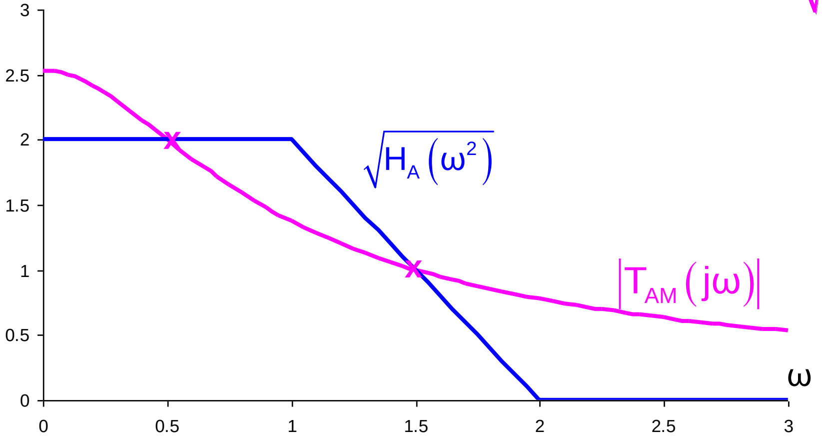
The approximation is reasonable but not too good

Collocation

Example:

$$H_A\left(\omega^2\right)=\frac{32/5}{1+\left(12/5\right)\omega^2}$$

$$T_{AM}\left(s\right)=\frac{\sqrt{8/3}}{s+\sqrt{5/12}}$$

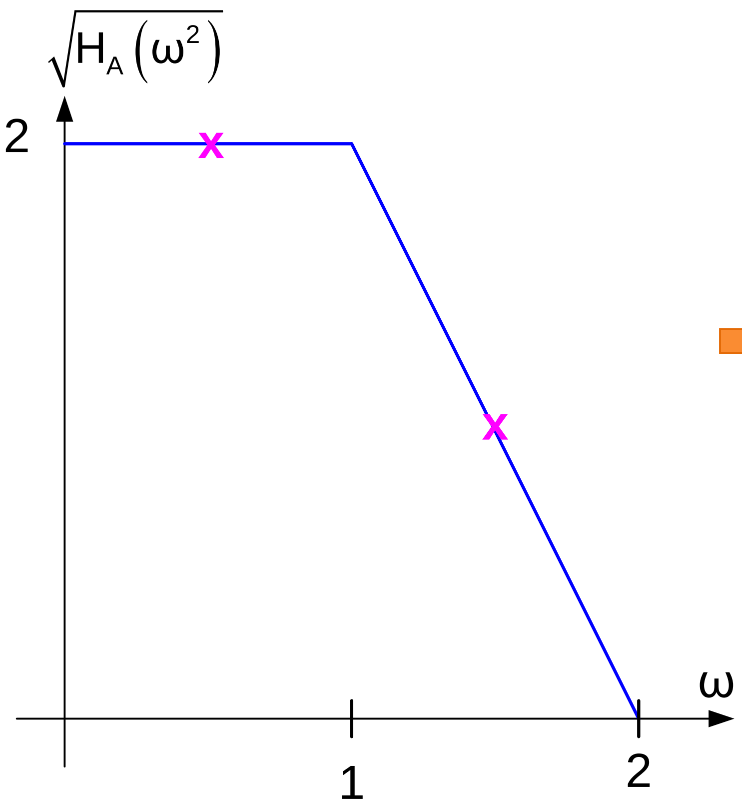


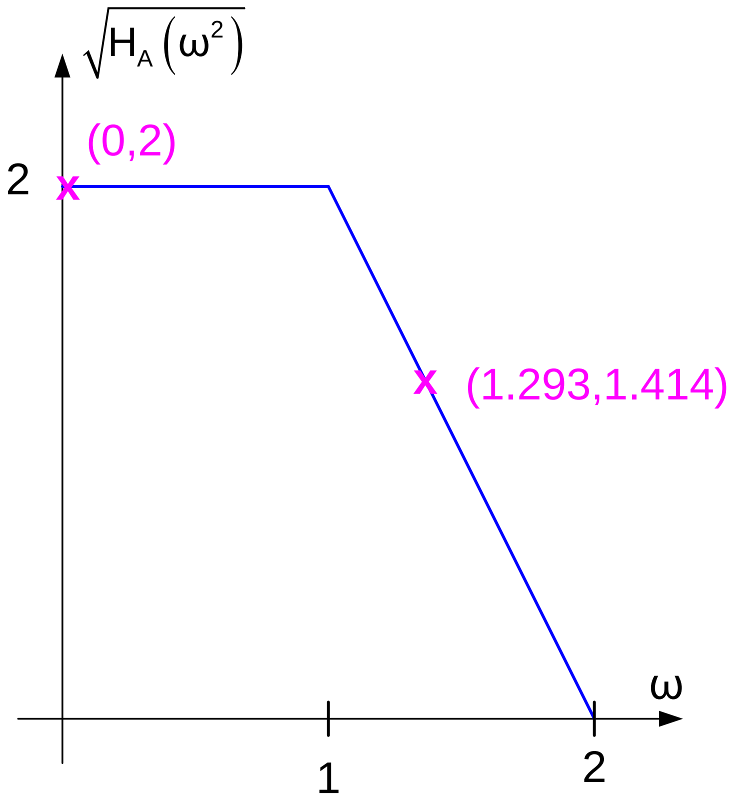
- The problem was critically constrained from a function viewpoint (two variables and two equations)
- Highly under-constrained as an approximation technique since the collocation points are also variables

Collocation

Example: same $H_A(\omega^2)$ but with different collocation points

$$H_A(\omega^2) = \frac{a_0}{1+b_1\omega^2}$$





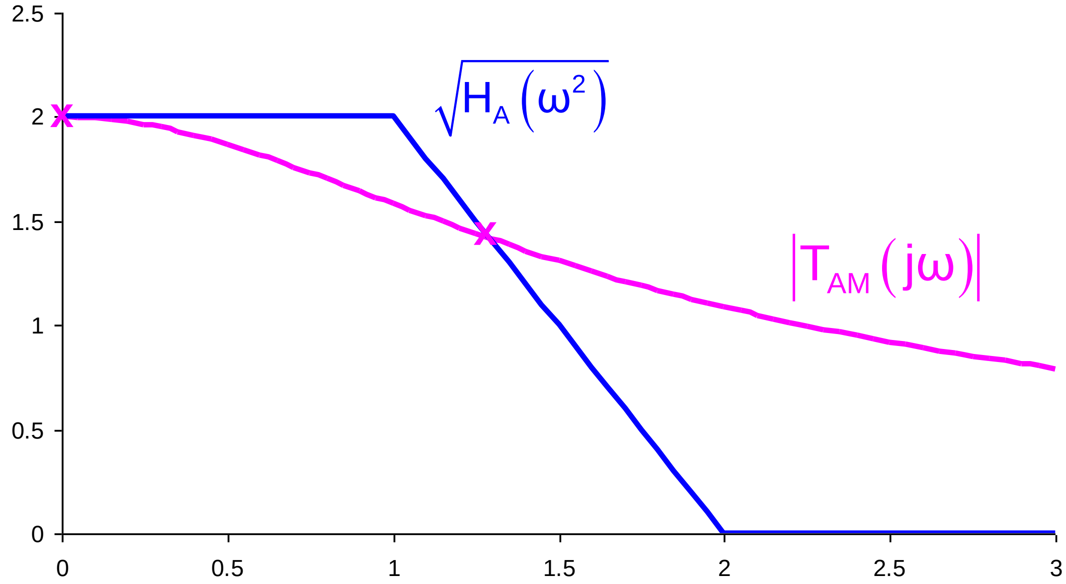
Collocation

Example: same $H_A\left(\omega^2\right)$ but with different collocation points

$$H_A\left(\omega^2\right)=\frac{a_0}{1+b_1\omega^2}$$

$$\left.\begin{array}{l} 4 = \dfrac{a_0}{1+b_1(0)^2} \\ 2 = \dfrac{a_0}{1+b_1(1.293)^2} \end{array}\right\} \Rightarrow \quad H_A\left(\omega^2\right)=\frac{4}{1+(.598)\omega^2}$$

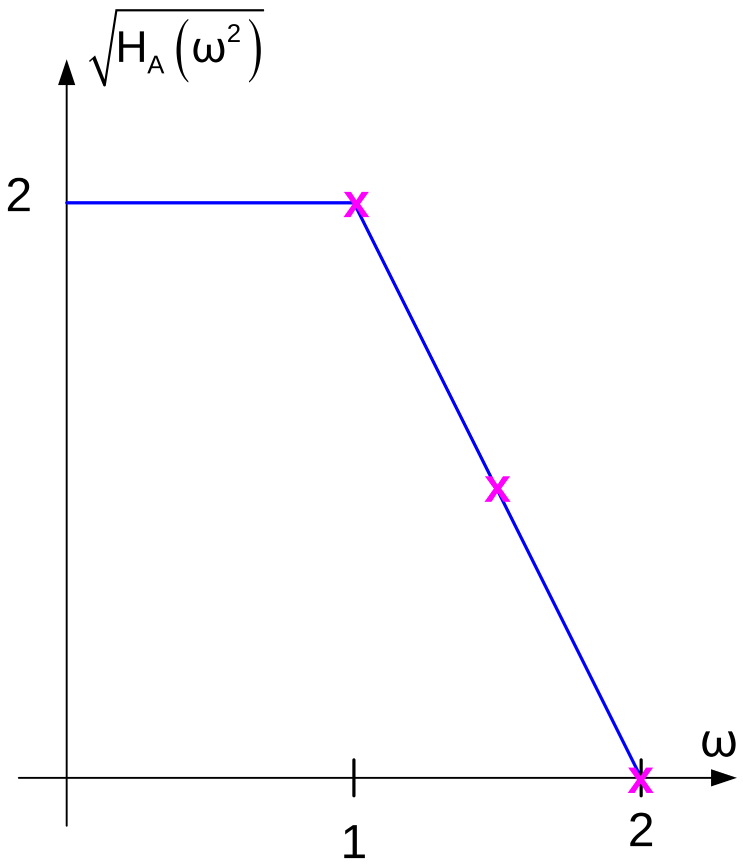
$$T_{AM}(s)=\frac{1.293}{s+1.293}$$



Choice of collocation points plays a big role on the approximation

Collocation

Example: same $H_A(\omega^2)$ but with different collocation points and different approximating function



$$H_A(\omega^2) = \frac{a_0 + a_1\omega^2}{1+b_1\omega^2}$$

$$\left.\begin{aligned} 4 &= \frac{a_0+a_1}{1+b_1} \\ 1 &= \frac{a_0+a_1(3/2)^2}{1+b_1(3/2)^2} \\ 0 &= \frac{a_0+a_1(4)}{1+b_1(4)} \end{aligned}\right\} \Rightarrow \quad H_A(\omega^2) = \frac{-80 + 20\omega^2}{1+ -16\omega^2}$$

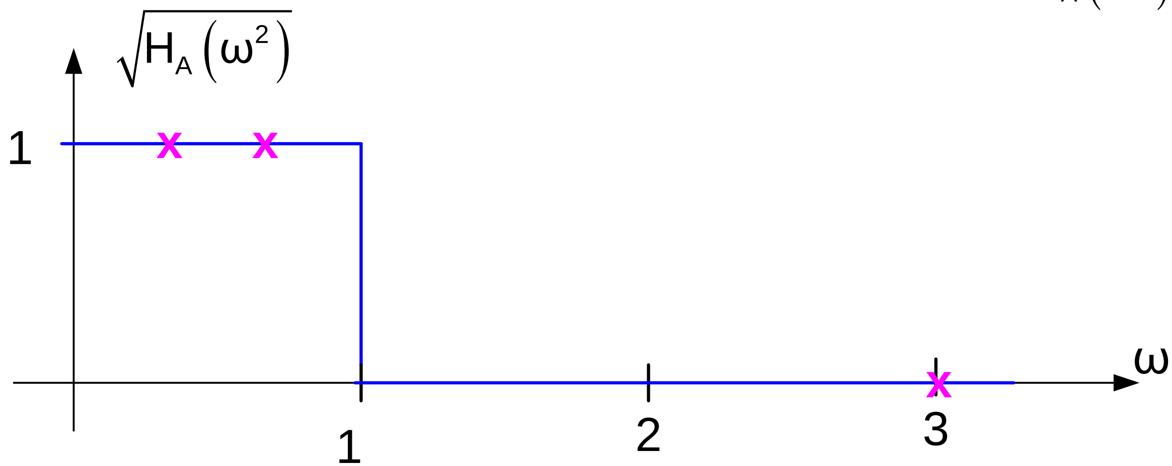
a_0=-80, a_1=20, b_1=-16

Inverse mapping does not exist because roots of odd multiplicity on real axis

Collocation

Example:

$$H_A\left(\omega^2\right) = \frac{a_0 + a_1\omega^2}{1+b_1\omega^2}$$



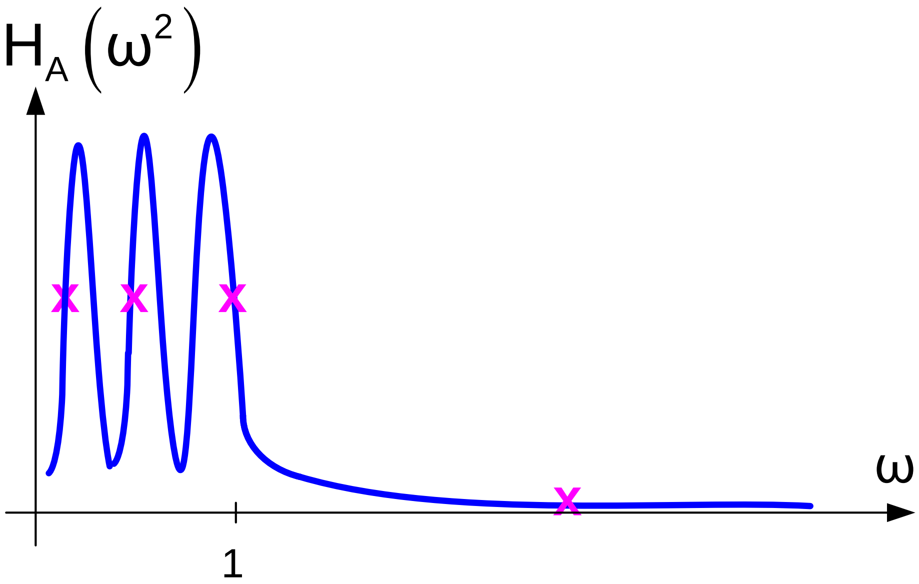
$$\left.\begin{aligned}
1 &= \frac{a_0 + a_1(1/9)}{1+b_1(1/9)} \\
1 &= \frac{a_0 + a_1(4/9)}{1+b_1(4/9)} \\
0 &= \frac{a_0 + a_1(9)}{1+b_1(9)}
\end{aligned}\right\} \Rightarrow \quad H_A\left(\omega^2\right) = \frac{1+(-27/243)\omega^2}{1+\ (-27/243)\omega^2}$$

$a_0=1$, $a_1=-27/243$, $b_1=-27/243$

- This solution is equal to 1 at all frequencies except ω=3 where it is undefined
- Thus there is no solution with these collocation points

Collocation

Example:



In some situations, collocation causes a lot of ripple between the collocation points

Collocation Observations

Fitting an approximating function to a set of data or points (collocation points)

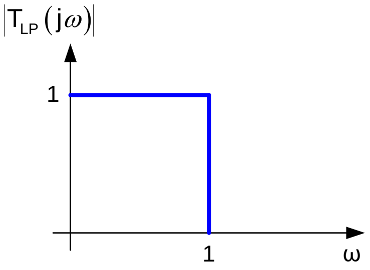
- Closed-form matrix solution for fitting to a rational fraction in ω^2
- Can be useful when somewhat nonstandard approximations are required
- Quite sensitive to collocation points
- Although function is critically constrained, since collocation points are variables, highly under constrained as an optimization approach
- Although fit will be perfect at collocation points, significant deviation can occur close to collocation points
- Inverse mapping to $T_A(s)$ may not exist
- Solution may not exist at specified collocation points

Collocation

What is the major contributor to the limitations observed with the collocation approach?

- Totally dependent upon the value of the desired response at a small but finite set of points (no consideration for anything else)
- Highly dependent upon value of approximating function at a single point or at a small number of points
- Highly dependent upon the collocation points

The Approximation Problem



Approach we will follow:

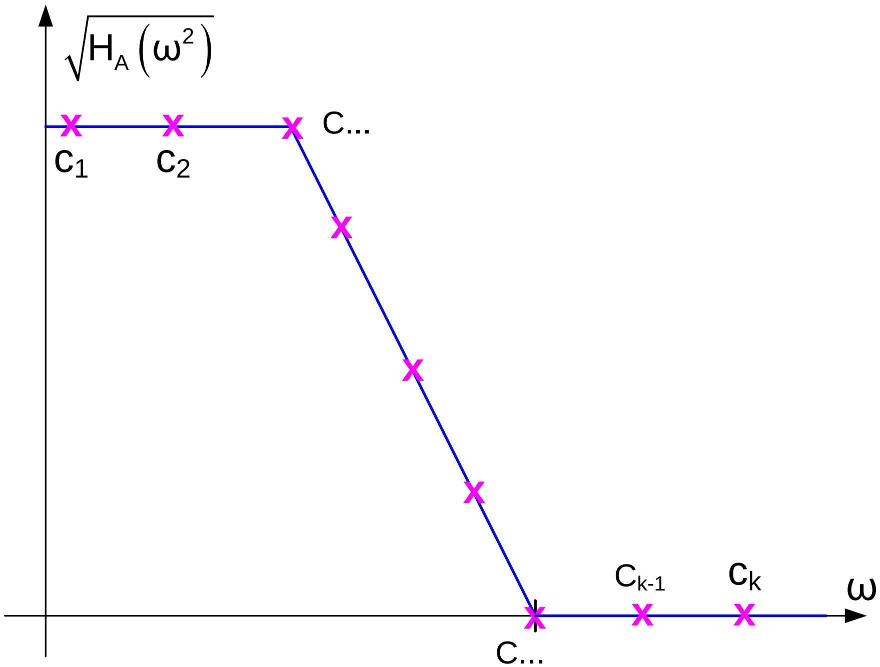
- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares
- Pade Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - $\rightarrow$ Butterworth (BW)
  - $\rightarrow$ Chebyschev (CC)
  - $\rightarrow$ Elliptic
  - $\rightarrow$ Thompson

Least Squares Approximation

To minimize the heavy dependence on a small number of points, will consider many points thus creating an over-constrained system

$$H_A\left(\omega^2\right) = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1+\sum_{i=1}^{n} b_i \omega^{2i}}$$

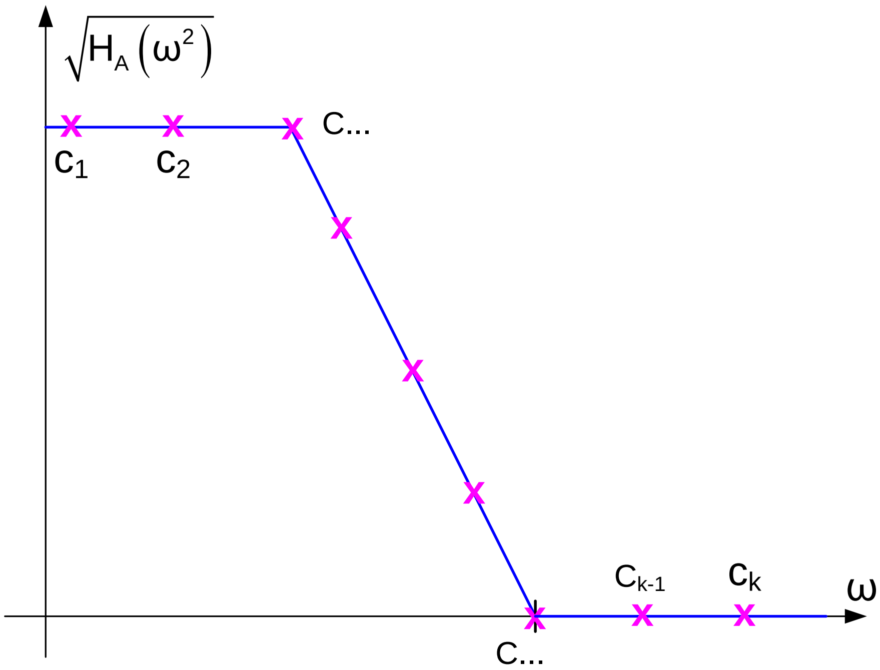
$k > m+n+1$



Approximating function can not be forced to go through all points
But, it can be “close” to all points in some sense

Least Squares Approximation

$$H_A\left(\omega^2\right) = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1+\sum_{i=1}^{n} b_i \omega^{2i}}$$



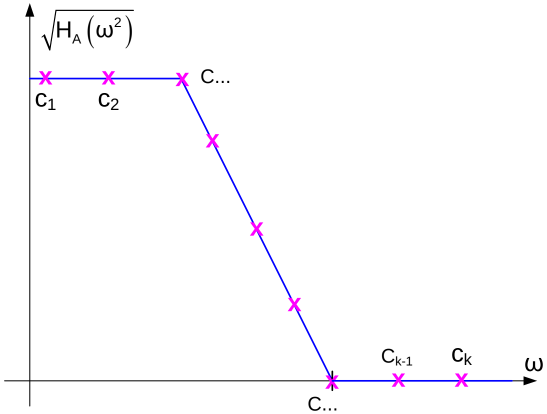
Define the error at point i by

$$\varepsilon_i = H_D\left(\omega_i\right) - H_A\left(\omega_i\right)$$

where $H_D(\omega_i)$ is the desired magnitude squared response at ω_i and where $H_A(\omega_i)$ is the magnitude squared response of the approximating function

Least Squares Approximation

$$H_A\left(\omega^2\right) = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1+\sum_{i=1}^{n} b_i \omega^{2i}}$$



$$\varepsilon_i = H_D\left(\omega_i\right) - H_A\left(\omega_i\right)$$

Goal is to minimize some metrics associated with ε_i at a large number of points

Some possible cost functions

$$C_1 = \sum_{i=1}^{N} \left|\varepsilon_i\right| \qquad C_2 = \sum_{i=1}^{N} \varepsilon_i^2 \qquad C_3 = \sum_{i=1}^{N} w_i\, \varepsilon_i^2$$

w_i a weighting function

- Reduces emphasis on individual points
- Some much better than others from performance viewpoint
- Some much better than others from computation viewpoint

Least Squares Approximation

$$H_A\left(\omega^2\right) = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1+\sum_{i=1}^{n} b_i \omega^{2i}}$$

$$\varepsilon_i = H_D\left(\omega_i\right) - H_A\left(\omega_i\right)$$

$$C_3 = \sum_{i=1}^{N} w_i\, \varepsilon_i^2$$

w_i a weighting function

Least Mean Square (LMS) based cost functions have minimums that can be analytically determined for some useful classes of approximating functions $H_A(\omega^2)$

Regression Analysis Review

Consider an nth order polynomial in x

$$F(x) = \sum_{k=0}^{n} a_k x^k$$

Consider N samples of a function $\tilde{F}(x)$

$$\hat{F}(x) = \left\langle \tilde{F}(x_i) \right\rangle_{i=1}^{N}$$

where the sampling coordinate variables are

$$X = \left\langle x_i \right\rangle_{i=1}^{N}$$

Define the summed square difference cost function as

$$C = \sum_{i=0}^{N} \left(F(x_i) - \tilde{F}(x_i) \right)^2$$

A standard regression analysis can be used to minimize C with respect to $\{a_0, a_1, \ldots a_n\}$

To do this, take the n+1 partials of C wrt the a_i variables

Regression Analysis Review

$$C = \sum_{i=0}^{N}\left(F(x_i) - \tilde{F}(x_i)\right)^2 \qquad F(x) = \sum_{k=0}^{n} a_k x^k$$

$$C = \sum_{i=0}^{N}\left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i)\right)^2$$

Taking the partial of C wrt each coefficient and setting to 0, we obtain the set of equations

$$\left.\begin{aligned}
\frac{\partial C}{\partial a_0} &= 2\sum_{i=0}^{N}\left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i)\right) = 0 \\
\frac{\partial C}{\partial a_1} &= 2\sum_{i=0}^{N} x_i^1\left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i)\right) = 0 \\
\frac{\partial C}{\partial a_2} &= 2\sum_{i=0}^{N} x_i^2\left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i)\right) = 0 \\
&\cdots \\
\frac{\partial C}{\partial a_n} &= 2\sum_{i=0}^{N} x_i^n\left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i)\right) = 0
\end{aligned}\right\}$$

This is linear in the a_ks.

$$X \bullet A = F$$

$$A = \begin{bmatrix} a_0 \\ a_1 \\ \ldots \\ a_n \end{bmatrix}$$

Solution is

$$A = X^{-1} \bullet F$$

Regression Analysis Review

A few details about regression analysis:

$$X \bullet A = F$$

$$A = X^{-1} \bullet F$$

$$X = \begin{bmatrix} N+1 & \sum_{i=0}^{N} X_i & \sum_{i=0}^{N} X_i^2 & \ldots & \sum_{i=0}^{N} X_i^n \\ \sum_{i=0}^{N} X_i & \sum_{i=0}^{N} X_i^2 & \ldots & & \sum_{i=0}^{N} X_i^{n+1} \\ \sum_{i=0}^{N} X_i^2 & \ldots. & & & \sum_{i=0}^{N} X_i^{n+2} \\ \ldots & & & & \\ \sum_{i=0}^{N} X_i^n & \sum_{i=0}^{N} X_i^{n+1} & \ldots & & \sum_{i=0}^{N} X_i^{2n} \end{bmatrix}$$

$$A = \begin{bmatrix} a_0 \\ a_1 \\ \ldots \\ a_n \end{bmatrix}$$

$$F = \begin{bmatrix} \sum_{i=0}^{N} \tilde{F}(x_i) \\ \sum_{i=0}^{N} x_i \tilde{F}(x_i) \\ \sum_{i=0}^{N} x_i^2 \tilde{F}(x_i) \\ \ldots \\ \sum_{i=0}^{N} x_i^n \tilde{F}(x_i) \end{bmatrix}$$

Regression Analysis Review

$$C = \sum_{i=0}^{N}\left(F(x_i) - \tilde{F}(x_i)\right)^2 \qquad F(x) = \sum_{k=0}^{n} a_k x^k$$

$$C = \sum_{i=0}^{N}\left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i)\right)^2$$

$$A = X^{-1} \bullet F$$

Observations about Regression Analysis:

- Closed form solution
- Requires inversion of a (n+1) dimensional square matrix
- Not highly sensitive to any single measurement
- Widely used for fitting a set of data to a polynomial model
- Points need not be uniformly distributed
- Adding weights does not complicate solution

This analysis was restricted to a polynomial – will see how applicable to a rational fraction !

Least Squares Approximations of Transfer Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

WLOG $b_0 = 1$

$$T(j\omega) = \frac{\left[\sum_{\substack{i=0 \\ i \text{ odd}}}^{m} (-1)^i a_i \omega^i\right] + \left[\sum_{\substack{i=0 \\ i \text{ even}}}^{m} (-1)^i a_i \omega^i\right] j}{\left[\sum_{\substack{i=0 \\ i \text{ odd}}}^{n} (-1)^i b_i \omega^i\right] + \left[\sum_{\substack{i=0 \\ i \text{ even}}}^{n} (-1)^i b_i \omega^i\right] j}$$

$$\left|T(j\omega)\right| = \sqrt{\frac{\left[\sum_{\substack{i=0 \\ i \text{ odd}}}^{m} (-1)^i a_i \omega^i\right]^2 + \left[\sum_{\substack{i=0 \\ i \text{ even}}}^{m} (-1)^i a_i \omega^i\right]^2}{\left[\sum_{\substack{i=0 \\ i \text{ odd}}}^{n} (-1)^i b_i \omega^i\right]^2 + \left[\sum_{\substack{i=0 \\ i \text{ even}}}^{n} (-1)^i b_i \omega^i\right]^2}}$$

$|T(j\omega)|$ is highly nonlinear in $\langle a_k \rangle$ and $\langle b_k \rangle$

Least Squares Approximations of Transfer Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

WLOG $b_0 = 1$

$$\left|T(j\omega)\right| = \sqrt{\frac{\left[\sum_{\substack{i=0\\ i\text{ odd}}}^{m} (-1)^i a_i \omega^i\right]^2 + \left[\sum_{\substack{i=0\\ i\text{ even}}}^{m} (-1)^i a_i \omega^i\right]^2}{\left[\sum_{\substack{i=0\\ i\text{ odd}}}^{n} (-1)^i b_i \omega^i\right]^2 + \left[\sum_{\substack{i=0\\ i\text{ even}}}^{n} (-1)^i b_i \omega^i\right]^2}}$$

Consider the natural cost function

$$C = \sum_{k=1}^{N} \left(\left|T(j\omega_k)\right| - \tilde{T}(\omega_k) \right)^2$$

$$\left.\begin{array}{c} \dfrac{\partial C}{\partial a_k} \\[2ex] \dfrac{\partial C}{\partial b_k} \end{array}\right\}$$

both are highly nonlinear in $<a_k>$ and $<b_k>$

Closed form solution for optimal values of $<a_k>$ and $<b_k>$ does not exist

Least Squares Approximations of Transfer Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} \qquad \text{WLOG } b_0 = 1 \qquad \xrightarrow{\text{Consider}} \qquad H_A\left(\omega^2\right) = \frac{\sum_{i=0}^{m} c_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}}$$

Consider the cost function

$$C = \sum_{k=1}^{N}\left(H_A\left(\omega_k^2\right) - \tilde{H}\left(\omega_k^2\right)\right)^2$$

What about the sets of equations $\left\langle \frac{\partial C}{\partial c_k} \right\rangle_{k=1}^{m}$ and $\left\langle \frac{\partial C}{\partial d_k} \right\rangle_{k=1}^{n}$

Rewriting the cost function

$$C = \sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m} c_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}} - \tilde{H}\left(\omega_k^2\right)\right)^2 \longrightarrow C = \sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m} c_i \omega^{2i} - \tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n} d_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}}\right)^2$$

$\left\langle \frac{\partial C}{\partial c_k} \right\rangle_{k=1}^{m}$ is linear in $\langle c_k \rangle$ $\qquad$ $\left\langle \frac{\partial C}{\partial d_k} \right\rangle_{k=1}^{n}$ is highly nonlinear in $\langle d_k \rangle$

Closed form solution for optimal values of $\langle c_k \rangle$ and $\langle d_k \rangle$ does not exist

Least Squares Approximations of Transfer Functions

$$H_A\left(\omega^2\right)=\frac{\sum_{i=0}^{m} c_i\omega^{2i}}{\sum_{i=0}^{n} d_i\omega^{2i}}$$

$$C=\sum_{k=1}^{N}\left(H_A\left(\omega_k^2\right)-\tilde{H}\left(\omega_k^2\right)\right)^2$$

$$C=\sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m} c_i\omega^{2i}-\tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n} d_i\omega^{2i}}{\sum_{i=0}^{n} d_i\omega^{2i}}\right)^2$$

$\left\langle \frac{\partial C}{\partial c_k} \right\rangle_{k=1}^{m}$ is linear in $<c_k>$ $\qquad$ $\left\langle \frac{\partial C}{\partial d_k} \right\rangle_{k=1}^{n}$ is highly nonlinear in $<d_k>$

But

if $<d_k>$ is fixed, optimal value of $<c_k>$ can be easily obtained

equivalently,

if poles of $H_A(\omega^2)$ are fixed, optimal value of zeros of $H_A(\omega^2)$ can be easily obtained

Is this observation useful?

Least Squares Approximations of Transfer Functions

$$C=\sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m}c_i\omega^{2i}-\tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n}d_i\omega^{2i}}{\sum_{i=0}^{n}d_i\omega^{2i}}\right)^2$$

if poles of $H_A(\omega^2)$ are fixed, optimal value of zeros of $H_A(\omega^2)$ can be easily obtained

$$C=\sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m}c_i\omega^{2i}-\tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n}d_i\omega^{2i}}{\sum_{i=0}^{n}\hat{d}_i\omega^{2i}}\right)^2$$

if poles of $H_A(\omega^2)$ are fixed in denominator of C, the partials of C wrt both $<c_k>$ and $<d_k>$ are linear in $<c_k>$ and $<d_k>$

Are these observations useful?

- Several optimization approaches can be derived from these observations
- Some will provide a LMS optimization of $H_A(\omega^2)$
- No guarantee that inverse mapping exists
- Some may provide a good approximation even though not truly LMS
- Others may not be useful

Least Squares Approximations of Transfer Functions

$$C = \sum_{k=1}^{N} \left(\frac{\sum_{i=0}^{m} c_i \omega^{2i} - \tilde{H}\left(\omega_k^2\right) \sum_{i=0}^{n} d_i \omega^{2i}}{\sum_{i=0}^{n} d_i \omega^{2i}} \right)^2$$

Possible uses of these observations (four algorithms)

1. Guess poles and obtain optimal zero locations
2. Start with a “good” T(s) obtained by any means and improve by selecting optimal zeros
3. Guess poles and then update estimates of both poles and zeros, use new estimate of poles and again update both zeros and poles, continue until convergence or stop after fixed number of iterations
4. Guess poles and obtain optimal zeros. Then invert function and cost and obtain optimal zeros (which are actually poles). Then invert again and obtain optimal zeros. Process can be repeated. - Weighting may be necessary to de-emphasize stop-band values when working with the inverse function

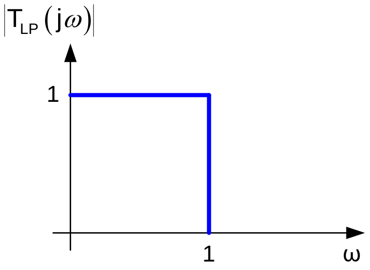
Least Squares Approximations of Transfer Functions

$$C=\sum_{k=1}^{N}\left(\frac{\sum_{i=0}^{m}c_i\omega^{2i}-\tilde{H}\left(\omega_k^2\right)\sum_{i=0}^{n}d_i\omega^{2i}}{\sum_{i=0}^{n}d_i\omega^{2i}}\right)^2$$

Comments/Observations about LMS approximations

1. **As with collocation, there is no guarantee that $T_A(s)$ can be obtained from $H_A(\omega^2)$**
2. **Closed-form analytical solutions exist for some useful mean square based cost functions**
3. **Any of the LMS cost functions discussed that have an analytical solution can have the terms weighted by a weight w_i. This weight will not change the functional form of the equations but will affect the fit**
4. **The best choice of sample frequencies is not obvious (both number and location)**
5. **The LMS cost function is not a natural indicator of filter performance**
6. **It is often used because more natural indicators are generally not mathematically tractable**
7. **The LMS approach may provide a good solution for some classes of applications but does not provide a universal solution**

The Approximation Problem



Approach we will follow:

- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares
- Pade' Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - $\rightarrow$ Butterworth (BW)
  - $\rightarrow$ Chebyschev (CC)
  - $\rightarrow$ Elliptic
  - $\rightarrow$ Thompson

Pade' Approximations

Henri Eugène Padé (December 17, 1863 – July 9, 1953) was a French mathematician, who is now remembered mainly for his development of approximation techniques for functions using rational functions.

The Pade' approximations were discussed in his doctoral dissertation in approximately 1890

(from Wikipedia)

Pade’ Approximations

Consider the polynomial

$$T_D(s) = \sum_{i=0}^{\infty} c_i s^i$$

Define the rational fraction Rm,n(s) by

$$R_{m,n}(s) = \frac{\sum_{i=0}^{m} a_i s^i}{1 + \sum_{i=1}^{n} b_i s^i} = \frac{A(s)}{B(s)}$$

The rational fraction $R_{m,n}(s)$ is said to be a (m,n)th order Pade’ approximation of $T_D(s)$ if $T_D(s)B(s)$ agrees with A(s) through the first m+n+1 powers of s

Note the Pade’ approximation applies to any polynomial with the argument being either real, complex, or even an operator s

Can operate directly on functions in the s-domain

Pade' Approximations

Example

$$T_D(s) = 1 + s + \left(\frac{1}{2!}\right)s^2 + \left(\frac{1}{3!}\right)s^3 + \dots$$

Determine $R_{2,3}(s)$

$$R_{2,3}(s) = \frac{a_0 + a_1 s + a_2 s^2}{1 + b_0 + b_1 s + b_2 s^2 + b_3 s^3} = \frac{A(s)}{B(s)}$$

setting

$$T_D(s)B(s) = A(s)$$

obtain

$$\left(1 + s + \left(\frac{1}{2!}\right)s^2 + \left(\frac{1}{3!}\right)s^3 + \dots\right)\left(1 + b_1 s + b_2 s^2 + b_3 s^3\right) = a_0 + a_1 s + a_2 s^2$$

Pade' Approximations

Example

$$T_D(s) = 1 + s + \left(\frac{1}{2!}\right)s^2 + \left(\frac{1}{3!}\right)s^3 + \dots$$

$$\left(1 + s + \left(\frac{1}{2!}\right)s^2 + \left(\frac{1}{3!}\right)s^3 + \dots\right)\left(1 + b_1 s + b_2 s^2 + b_3 s^3\right) = a_0 + a_1 s + a_2 s^2$$

$$\left.\begin{aligned}
a_0 &= 1 \\
a_1 &= 1 + b_1 \\
a_2 &= b_1 + b_2 + \frac{1}{2!} \\
0 &= b_2 + b_3 + \frac{b_1}{2} + \frac{1}{6} \\
0 &= b_3 + \frac{b_2}{2} + \frac{b_1}{6} + \frac{1}{24} \\
0 &= \frac{b_3}{2} + \frac{b_2}{6} + \frac{b_1}{24} + \frac{1}{5!}
\end{aligned}\right\} \Rightarrow \quad
\begin{aligned}
b_1 &= -.6 \\
b_2 &= .15 \\
b_3 &= -.01666 \\
a_0 &= 1 \\
a_1 &= 0.4 \\
a_2 &= .05
\end{aligned}$$

Pade' Approximations

Example

$$T(s) = \frac{1 + 0.4\,s + 0.05s^2}{1 - 0.6s + 0.15s^2 - 0.01\overline{6}\,s^3}$$

$b_1 = -.6$

$b_2 = .15$

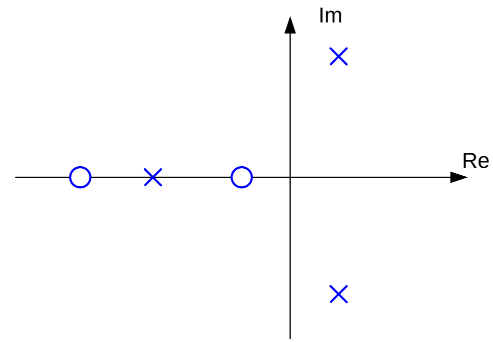
$b_3 = -.01666$

$a_0 = 1$

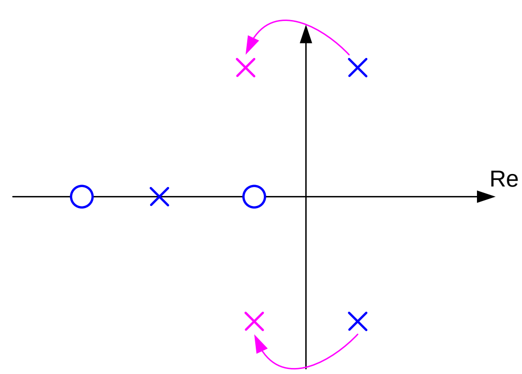
$a_1 = 0.4$

$a_2 = .05$

T(s) has a pair of cc poles in the RHP and is thus unstable!



Poles can be reflected back into the LHP to obtain stability and maintain magnitude response



Pade' Approximations

If $T_A(s)$ is an all pole approximation, then the Pade' approximation of $1/T_A(s)$ is the reciprocal of the Pade' approximation of $T_A(s)$

Pade' approximations can be made for either $T_A(s)$ or $H_A(\omega^2)$.

$$\begin{tikzcd}
T_A\left(s\right) \arrow[r, bend left] \arrow[d, bend right] & H_A\left(\omega^2\right) \arrow[d, bend left] \\
R_{m,n}\left(s\right) \arrow[r, leftrightarrow, "?"] & \tilde{R}_{m,n}\left(\omega^2\right)
\end{tikzcd}$$

Is it better to do Pade' approximations of $T_A(s)$ or $H_A(\omega^2)$?

What relationship, if any, exists between $R_{m,n}\left(s\right)$ and $\tilde{R}_{m,n}\left(s\right)$?

Pade' Approximations

- Useful for order reduction of all-pole or all-zero approximations

- Can map an all-zero approximation to a realizable rational fraction in the s-domain

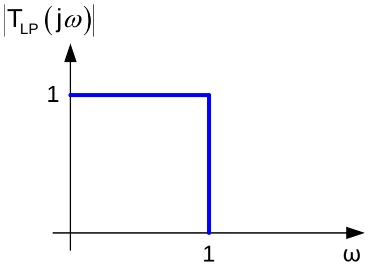
- Can extend concept to provide order reduction of higher-order rational fraction approximations

- Can always maintain stability or even minimum phase by reflecting any RHP roots back into the LHP

- Pade' approximation is heuristic (no metrics associated with the approach)

- No guarantees about how good the approximations will be

The Approximation Problem



Approach we will follow:

- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares
- Pade' Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - $\rightarrow$ Butterworth (BW)
  - $\rightarrow$ Chebyschev (CC)
  - $\rightarrow$ Elliptic
  - $\rightarrow$ Thompson

Other Analytical Approximations

- Numerous analytical strategies have been proposed over the years for realizing a filter
- Some focus on other characteristics (phase, time-domain response, group delay)
- Almost all based upon real function approximations
- Remember – inverse mapping must exist if a useful function T(s) is to be obtained

Approximations

- Magnitude Squared Approximating Functions – $H_A(\omega^2)$
- Inverse Transform - $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares Approximations
- Pade Approximations
- Other Analytical Optimizations
- Numerical Optimization
- Canonical Approximations
  - Butterworth
  - Chebyschev
  - Elliptic
  - Bessel
  - Thompson

Numerical Optimization

- Optimization algorithms can be used to obtain approximations in either the s-domain or the real domain
- The optimization problem often has a large number of degrees of freedom (m+n+1)

$$T(s)=\frac{\sum_{k=0}^{m} a_k s^k}{1+\sum_{k=0}^{n} b_k s^k}$$

- Need a good cost function to obtain good approximation
- Can work on either coefficient domain or root domain or other domains
- Rational fraction approximations inherently vulnerable to local minimums
- Can get very good results

End of Lecture 8